LECTURE 2: CLASSICAL DEFORMATION THEORY IN ALGEBRAIC GEOMETRY

1. Pre-deformation functors

Yesterday we saw some first examples of deformation problems. Today we will focus on problems coming from algebraic geometry, and the classical deformation theory thereof.

Recall that algebraic geometry (roughly) is the study of solutions of systems of polynomial equations, which form *algebraic varieties* or, more generally, *schemes*.

We will mostly think in terms of the *functor of points*. Let X be a scheme over a base ring R. Consider the functor

$$\begin{aligned} h_X : \textsc{Schemes}_R^{op} &\longrightarrow \textsc{Sets} \\ Y &\longmapsto \mathrm{Hom}_{\textsc{Schemes}_R}(Y, X). \end{aligned}$$

By the Yoneda embedding, the functor $h : \textsc{Schemes}_R \to \mathrm{Fun}(\textsc{Schemes}_R^{op}, \textsc{Sets})$ is fully faithful, so we can study schemes by considering their functor of points. In fact, is enough to restrict h_X to affine schemes, i.e. $h_X : \textsc{CAlg}_R \to \textsc{Sets}$, and from now on, we will always consider such functors.

We start by deforming a point in a scheme using the functor of point description.

Example 1.1. Consider the (affine group) scheme $X = \mathrm{SL}_2$, given by

$$\textsc{CAlg}_R \ni A \longmapsto \mathrm{SL}_2(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in A, ad - bc = 1 \right\}$$

To find its tangent space, we want to find first order deformations of $id = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. These are matrices

$$\begin{pmatrix} 1 + \varepsilon a' & \varepsilon b' \\ \varepsilon c' & 1 + \varepsilon d' \end{pmatrix}$$

such that $(1 + \varepsilon a')(1 + \varepsilon d') - \varepsilon b' \varepsilon c' = 1$, which amounts to $a' + d' = 0$,

$$T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathrm{SL}_2 = \mathfrak{sl}_2.$$

More generally, instead of working with the dual numbers $\mathbb{K}[\varepsilon]/\varepsilon^2$ we will work with *Artinian algebras*, which encode *fat points*. That is, $S = \operatorname{Spec} A$ is a $\mathbb{K}$-scheme of finite type such that $S_{red} = \operatorname{Spec} \mathbb{K}$, i.e. the underlying topological space of S is just a point, but with interesting algebra of functions:

Definition 1.2. Let A be a local $\mathbb{K}$-algebra with residue field $\mathbb{K}$. We call A Artinian if one of the following equivalent conditions are satisfied:

(1) every descending chain of ideals stabilizes,

(2) A is finite dimensional as a $\mathbb{K}$-vector space,

(3) the maximal ideal of A is finitely generated and nilpotent.

The map $\mathbb{K} \to A \to A/\mathfrak{m}_A \cong \mathbb{K}$ exhibits $\mathbb{K}$ as a retract of A and $A \cong \mathbb{K} \oplus \mathfrak{m}_A$ as vector spaces.

Example 1.3. Let X be a locally Noetherian scheme over $\mathbb{K}$, and p a point in X, i.e. $p : \operatorname{Spec} \mathbb{K} \to \operatorname{Spec} R \subset X$, which in turn is equivalent to a map $p : R \to \mathbb{K}$. For an Artinian algebra A, an A-deformation is a map $\operatorname{Spec} A \to X$ which maps the (unique) closed point to p. This is equivalent to giving a map $\phi : R \to A$ such that $\bar{\phi} : R \to A/\mathfrak{m}_A \cong \mathbb{K}$ is p. We can encode this as a functor

$$\begin{aligned} \widehat{X}_p : \textsc{Art}_{\mathbb{K}} &\longrightarrow \textsc{Set} \\ A &\longmapsto \{\phi : R \to A \text{ s.t. } p = \bar{\phi} : R \to A/\mathfrak{m}_A \cong \mathbb{K}\} \end{aligned}$$

Note that $\widehat{X}_p(\mathbb{K}) = *$.

This leads to the first attempt at formalizing deformation problems, and we'll see later what conditions we might want to add:

Definition 1.4. A *pre-deformation functor*[1] is a functor

$$D : \textsc{Art}_{\mathbb{K}} \longrightarrow \textsc{Sets}$$

such that $D(\mathbb{K})$ is a single point. A *morphism* of pre-deformation functors is a natural transformation between the functors. Note that it is an isomorphism if it is bijective on each element in $\textsc{Art}_{\mathbb{K}}$.

Example 1.3 shows that every algebra R determines a pre-deformation functor. However, several algebras might give the same one. In fact, to avoid this ambiguity, we will consider the following ones:

Definition 1.5. Let $\textsc{CLoc}_{\mathbb{K}}$ be the category of complete[2] local $\mathbb{K}$-algebras with residue field $\mathbb{K}$. For $R \in \textsc{CLoc}_{\mathbb{K}}$, define the deformation functor

$$h_R(A) = \operatorname{Hom}_{\mathbb{K}}(R, A).$$

A pre-deformation functor is called *pro-representable* if it is isomorphic to h_R for some $R \in \textsc{CLoc}_{\mathbb{K}}$.

In Example 1.3, one can show that $\widehat{X}_p = \textsc{CAlg}_{\mathbb{K}}(\widehat{\mathcal{O}}_{X,p} = R_p, -)$, i.e. it is pro-representable.

Remark 1.6. If $R \in \textsc{CLoc}_{\mathbb{K}}$ is regular, then $R = \widehat{R}$ is a formal power series ring $\mathbb{K}[\![x_1, \ldots, x_d]\!]$. This explains why we call this *formal* deformation theory.

Example 1.7. Generalizing the Example 1.3, we could start with something more general than a scheme such as an arbitrary functor

$$F : \textsc{CAlg}_R \to \textsc{Sets},$$

which one can think of as a *classical moduli problem*. For example, recall the moduli problem of associative structures on a fixed vector space Owen introduced in the first lecture. Another example is that of the moduli of isomorphism classes of elliptic curves. Given a point $p \in F(\operatorname{Spec} \mathbb{K})$, we obtain a pre-deformation functor by

$$\begin{aligned} D_{F,p} : \textsc{Art}_{\mathbb{K}} &\longrightarrow \textsc{Sets} \\ A &\longmapsto \{\alpha \in F(\operatorname{Spec} A) : \alpha|_{\operatorname{Spec}(A/\mathfrak{m}_A)} = p\}. \end{aligned}$$

[1]The terminology is not consistent in the literature: sometimes this is called a functor of Artinian rings or simply a deformation problem. We will use the later term for pre-deformation functors with certain properties to be consistent with the terminology in [1].

[2]This means that $R = \widehat{R} := \lim R/\mathfrak{m}_R^N$.

Unpacking the condition on α, this is equivalent to defining $D_{F,p}(A)$ to be the pullback

$$\begin{array}{ccc} D_{F,p}(A) = D(A) \times_{D(\mathbb{K})} \{p\} & \longrightarrow & D(A) \\ \downarrow & & \downarrow \\ \{p\} & \longrightarrow & * = D(\mathbb{K}) \end{array}$$

2. Deformation functors

As we saw in the examples, the unique point in $D(\mathbb{K})$ should be thought of as the object we want to deform. The set $D(A)$ should be thought of the isomorphism classes of A-deformations of that object. Note that for any Artinian $\mathbb{K}$-algebra A, there is a map $\mathbb{K} \to A$ given by the unit, which picks out an element $* = D(\mathbb{K}) \to D(A)$, which is the trivial deformation.

Now we can ask the following questions for given $A \in \mathrm{Art}_{\mathbb{K}}$:

(1) Can the object $* \in D(\mathbb{K})$ be deformed over A in a non-trivial way, i.e. whether $D(A)$ contains more than the trivial deformation?

(2) If so, how many there are?

This can be analyzed by studying the surjective morphism $A \to \mathbb{K}$, which, since A is Artinian, can be be decomposed into a chain of surjective morphisms

$$A \to A/\mathfrak{m}_A^n \to \cdots \to A/\mathfrak{m}_A^2 \to A/\mathfrak{m}_A \cong \mathbb{K}. \tag{1}$$

Thus, we will study how the deformations for varying A can be related to each other: Let $\sigma : B \to A$ be a surjective morphism of Artinian $\mathbb{K}$-algebras.

(1') Which elements $\alpha \in D(A)$ lift, i.e. $\exists \beta \in D(\beta)$ such that $D(\sigma)(\beta) = \alpha$?

(2') How many lifts are there?

In other words, we want to study the image of the map $D(\sigma)$ and its fiber at a point.

It is a fact that the category $\mathrm{Art}_{\mathbb{K}}$ is closed under fiber products. We will require deformation functors to respect this property, which will allow us to analyze the maps $D(\sigma)$, as we will see later. In examples, we will see that this is a natural condition to require. It should remind you of Mayer-Vietoris sequences.

Definition 2.1. A *deformation functor*[3] is a pre-deformation functor $D : \mathrm{Art}_{\mathbb{K}} \to \mathrm{Sets}$ such that for every fiber product

$$\begin{array}{ccc} B \times_A C & \longrightarrow & B \\ \downarrow & & \downarrow \sigma \\ C & \longrightarrow & A \end{array}$$

the induced map

$$D(B \times_A C) \to D(B) \times_{D(A)} D(C)$$

(D1) is surjective if $\sigma : B \to A$ is surjective,

(D2) is bijective if $A = \mathbb{K}$.

[3]As with pre-deformation functors, there are variations on this definition, see the references for more details.

Example 2.2. For any R the functor h_R is a deformation functor, since in this case, the maps

$$h_R(B \times_A C) \to h_R(B) \times_{h_R(A)} h_R(C)$$

are even bijective.

Generalizing the composition (1), any surjective morphism $\sigma : B \to A$ can be written as a finite composition of *small morphisms*. It will be enough to check the condition for being a deformation functor for squares where σ is small.

Definition 2.3. A morphism $\sigma : B \to A$ is called *small* if it is a surjection and its kernel M satisfies $M \cdot \mathfrak{m}_B = 0$. A *small extension*

$$0 \to M \to B \xrightarrow{\sigma} A \to 0$$

is an exact sequence in $\mathrm{ART}_{\mathbb{K}}$ such that σ is small. If furthermore M is principal, we call the morphism or extension *elementary*.

Remark 2.4. Note that B Artinian implies that $\mathfrak{m}_B^N = 0$ for some N. The above condition means that $M \subset \mathfrak{m}_B^{N-1}$.

Example 2.5. The exact sequence

$$0 \to (\varepsilon) \cong \mathbb{K} \xrightarrow{\cdot\varepsilon} \mathbb{K}[\varepsilon]/\varepsilon^2 \longrightarrow \mathbb{K} \cong \mathbb{K}[\varepsilon]/\varepsilon \to 0 \tag{2}$$

is a small extension. More generally,

$$0 \to \mathbb{K} \longrightarrow \mathbb{K}[\varepsilon]/\varepsilon^n \longrightarrow \mathbb{K} \cong \mathbb{K}[\varepsilon]/\varepsilon^{n-1} \to 0$$

is a small extension for any $n > 1$.

Being a deformation functor allows us to give answers to the questions we asked above. The second questions (2) and (2') are answered by studying the *tangent space*:

Theorem 2.6. *([2], Lemma 2.12 in [3]) If $D : \mathrm{ART}_{\mathbb{K}} \to \mathrm{SETS}$ is a deformation functor, then*

(1) $T_D = D(\mathbb{K}[\varepsilon]/\varepsilon^2)$ has a natural structure of a vector space,

(2) if M is a finite dimensional vector space, then there is a canonical bijection $D(\mathbb{K} \oplus M) \cong T_D \otimes M$,

(3) for any small extension

$$0 \to M \to B \to A \to 0,$$

there is a canonical transitive action of $T_D \otimes M$ on the fibers of $D(\sigma) : D(B) \to D(A)$ functorial in the extension.

Example 2.7. For any R, the tangent space to h_R is $T_R = (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$.

Proof. (Sketch)

(1) Multiplication by scalars on $T_D = D(\mathbb{K}[\varepsilon]/\varepsilon^2)$ is given by the image under D of $\mathbb{K}[\varepsilon]/\varepsilon^2 \xrightarrow{\cdot\lambda} \mathbb{K}[\varepsilon]/\varepsilon^2$, and the additive structure is given by the image of $\mathbb{K}[\varepsilon_1, \varepsilon_2]/(\varepsilon_1, \varepsilon_2)^2 \cong \mathbb{K}[\varepsilon]/\varepsilon^2 \times \mathbb{K}[\varepsilon]/\varepsilon^2 \to \mathbb{K}[\varepsilon]/\varepsilon^2$.

(2) This proven by induction on $n = \dim M$: if $n = 1$, this is clear; if $n > 1$, write $M = M_1 \oplus M_2$ and consider the split-zero extensions $\mathbb{K} \oplus M_1$ and $\mathbb{K} \oplus M_2$. Then by (D2), $D(\mathbb{K} \oplus M) \cong D(\mathbb{K} \oplus M_1) \times D(\mathbb{K} \oplus M_2)$.

(3) **Case 1:** If the extension is split, $B = A \times_{\mathbb{K}} (\mathbb{K} \oplus M)$, i.e. we have a pullback diagram

$$\begin{array}{ccc} B = A \times_{\mathbb{K}} (\mathbb{K} \oplus M) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{K} \oplus M & \longrightarrow & \mathbb{K} \end{array}$$

which gives a surjection $D(B) \twoheadrightarrow D(A) \times (T_F \otimes M)$.
Case 2: If the extension is not split, consider $C = B \times_A B = B \oplus M$. The two pullback diagrams

$$\begin{array}{ccc} C = B \oplus M & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathbb{K} \oplus M & \longrightarrow & \mathbb{K} \end{array} \qquad \begin{array}{ccc} C = B \times_A B & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

given a surjection $D(B) \times (T_D \otimes M) \twoheadrightarrow D(B) \times_{D(A)} D(B)$ which is the action.

□

Answering the first questions (1) and (1') is more difficult. It requires studying *obstruction spaces*.

Definition 2.8. Let $D : \textsc{Art}_K \to \textsc{Sets}$ be a pre-deformation functor. An *obstruction theory* (V, v_e) for D is a vector space V and a map $v_e : D(A) \to V \otimes M$ for every small extension

$$0 \to M \to B \to A \to 0$$

such that if $A = \mathbb{K}$, then $v_e(* = D(A)) = 0$ and the maps v_e are functorial in small extensions.

A *morphism of obstruction theories* $(V, v_e) \to (W, w_e)$ is a morphism of vector spaces $f : V \to W$ such that for every small extension e we have that $w_e = (f \otimes id) \circ v_e$.

Lemma 2.9. *Let $D : \textsc{Art}_K \to \textsc{Sets}$ be a pre-deformation functor with obstruction theory (V, v_e) and let $0 \to M \to B \to A \to 0$ be a small extension. If an element $x \in D(A)$ lifts to $D(B)$, then $v_e(x) = 0$.*

However, we are interested in the converse statement!

Definition 2.10. An obstruction theory (V, v_e) for D is *complete* if the above Lemma gives a characterization of lifting elements, i.e. if an element $x \in D(A)$ can be lifted to $D(B)$ iff $v_e(x) = 0$.

In fact, every deformation functor admits a complete obstruction theory which moreover is a universal one, meaning that is initial in the category of obstruction theories.

Theorem 2.11. *([3]) Let $D : \textsc{Art}_K \to \textsc{Sets}$ be a deformation functor. Then there is an obstruction theory (O_D, od_e) which is universal and complete. We call O_D the* obstruction space *of D.*

Example 2.12. For any R, to find the obstruction space of h_R, we need to recall the following fact from commutative algebra: for $R \in \textsc{CLoc}_{\mathbb{K}}$, let $b_1, \ldots, b_d$ be a basis of the cotangent space $\mathfrak{m}_R/\mathfrak{m}_R^2$. Then the homomorphism

$$S = \mathbb{K}[\![x_1, \ldots, x_d]\!] \longrightarrow \widehat{R} = R, \quad x_i \longmapsto b_i$$

is surjective. Its kernel J lies in $\mathfrak{m}_S = (x_1, \dots, x_d)$. For any $R \in \mathrm{CLoc}_{\mathbb{K}}$ the pre-deformation functor h_R the obstruction space to D is $O_{h_R} = (J/\mathfrak{m}_S J)^\vee$.

Summarizing, for every deformation functor D we have a *tangent-obstruction theory*, i.e. for every small extension $0 \to M \to B \to A$, there is an exact sequence of sets

$$T_D \otimes_{\mathbb{K}} M \longrightarrow D(B) \longrightarrow D(A) \xrightarrow{ob} O_D \otimes_{\mathbb{K}} M,$$

and, if $A = \mathbb{K}$, this exact sequence can be extended to

$$0 \longrightarrow T_D \otimes_{\mathbb{K}} M \longrightarrow D(B) \longrightarrow D(\mathbb{K}) = * \longrightarrow O_D \otimes_{\mathbb{K}} M,$$

and these sequences are functorial in small extensions.

3. Pro-representable hulls

Even though though many functors in deformation theory are not representable, it turns out that deformation functors with finite dimensional tangent space can be approximated by pro-representable ones.

Definition 3.1. A morphism of pre-deformation functors $f : D_1 \to D_2$ is *smooth* if for every small extension $0 \to M \to B \to A$ the canonical map $D_1(B) \to D_2(B) \times_{D_2(A)} D_1(A)$ is surjective, i.e. an element $\alpha \in D_1(A)$ can be lifted to $D_1(B)$ if and only if $f(\alpha) \in D_2(A)$ can be lifted to $D_2(B)$.

Definition 3.2. Let D be a pre-deformation functor. A *pro-representable hull* for D is a deformation functor h_R for some $R \in \mathrm{CLoc}_{\mathbb{K}}$ together with a smooth morphism $h_R \to D$ which is bijective on the tangents, i.e. $T_R = h_R(\mathbb{K}[\varepsilon]/\varepsilon^2) \to T_D = D(\mathbb{K}[\varepsilon]/\varepsilon^2)$ is bijective.

Remark 3.3. A morphism of schemes $f : X \to Y$ is smooth at a point p iff $D_{X,p} \to D_{Y,f(p)}$ is smooth. Moreover, it is étale if furthermore it is an isomorphism on the tangent spaces.

Theorem 3.4. *(Schlessinger) Let $D : \mathrm{Art}_{\mathbb{K}} \to \mathrm{Sets}$ be a deformation functor with finite dimensional tangent space T_D. Then there is a pro-representable hull for D.*

4. Example: Deformation of smooth varieties

Consider the deformation problem of deforming a smooth variety X_0 over a field $\mathbb{K}$,

$$Def_{X_0}(A) = \{(X, f) : X \text{ flat scheme over } A, f : X \times_A \mathbb{K} \xrightarrow{\cong} X_0\}/\{\text{iso}\}.$$

We will follow [4] and compute its tangent and obstruction spaces. Recall that since X_0 is smooth, it's tangent sheaf is given by $T_{X_0} = \mathcal{H}om(\Omega^1_{X_0/\mathbb{K}}, \mathcal{O}_{X_0})$. The trivial first-order deformation of X_0 is $X = X_0[\varepsilon] = X_0 \times_{\mathbb{K}} \mathbb{K}[\varepsilon]/\varepsilon^2$. An *infinitesimal automorphism* of X_0 is an automorphism of $X_0[\varepsilon]$ which restricts to the identity on X_0. We will use the fact that the sheaf of infinitesimal automorphisms of a smooth variety is the tangent sheaf[4].

Theorem 4.1. *Let X_0 be a smooth variety over $\mathbb{K}$.*

1. *The tangent space of Def_{X_0} is $T_{X_0} = H^1(X_0, T_{X_0})$.*
2. *The obstruction space of Def_{X_0} is $O_{X_0} = H^2(X_0, T_{X_0})$.*

[4]Note that this is the analog of a well-known fact in differential geometry: infinitesimal automorphisms come from vector fields which are the direction for each point to flow along.

Remark 4.2. We can also interpret $H^0(X_0, T_{X_0})$, namely as the tangent space to automorphisms of X_0, and this acts on first-order deformations. This gives a geometric interpretation of H^0, H^1 and furthermore, the proof of (1) is rather conceptual using classifying torsors. However, (2) is rather computational rather than conceptual, and does not explain the whole cohomology group (but rather only the image of *ob*).

Proof. (Sketch) The first step in the proof of this theorem is to show that locally, i.e. on affine opens, all first-order deformations are trivial. This follows from an exercise in commutative algebra. Then we globalize: let $\{U_i\}_{i\in I}$ be an open affine cover of X_0. For any first-order deformation X_1, the restriction $X_1|_{U_i}$ is isomorphic to the trivial deformation for each i. Choose such trivializations $\varphi_i : X_1|_{U_i} \xrightarrow{\cong} U_i[\varepsilon]$. On overlaps $U_{i,j} = U_i \cap U_j$, these trivializations give maps

$$\varphi_{i,j} : U_{i,j}[\varepsilon] \xrightarrow{\varphi_i^{-1}} X_1|_{U_{i,j}} \xrightarrow{\varphi_j} U_{i,j}[\varepsilon],$$

which are the identity when restricted to $U_{i,j} = U_{i,j}[\varepsilon] \times_{\mathbb{K}[\varepsilon]/\varepsilon^2} \mathbb{K}$ and hence are infinitesimal automorphisms of $U_{i,j}$, i.e. sections os T_{X_0}. Furthermore, they satisfy the cocycle condition

$$\varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k}.$$

Conversely, given a cocycle in $\check{C}^1(X_0, T_{X_0})$, we can glue to obtain a first order deformation X_1. It remains to check when two cocycles give the same deformation, which turns out to de exactly given when their difference is a coboundary.

To find the obstruction space, one first needs that on affine opens, all A-deformations for any Artinian algebra A are trivial, which is a generalization of the first step of the proof. Let $\sigma : B \to A$ be a surjection, and assume that its kernel I is 1-dimensional. Given a deformation over $\operatorname{Spec} A$, locally on the affine opens U_i it is trivial, and can uniquely be lifted to deformations over $\operatorname{Spec} B$, which moreover are isomorphic on overlaps $U_{i,j}$, and these isomorphisms satisfy the cocycle condition. Conversely, given isomorphisms $\varphi_{i,j}$ of $U_{i,j}$ satisfying the cocycle condition when restricted to A, the question is whether we can glue them to obtain a deformation over B. In general, the answer will be no, and the failure $\varphi_{i,j} \circ \varphi_{j,k} - \varphi_{i,k}$ are infinitesimal automorphisms of $U_{i,j,k}$, i.e. sections of T_{X_0} which form a 2-cocyle. Moreover, their ambiguity is given precisely by a 2-coboundary, thus we obtain a well-defined element in $\check{H}^2(X_0, T_{X_0})$. □

5. Two more examples

Example 5.1. (Deformation of vector bundles) Consider the deformation problem of deforming a vector bundle $\mathcal{E}$ on a variety X, i.e. a locally free sheaf,

$$Def_{\mathcal{E}}(A) = \{(\mathcal{F}, f) : \mathcal{F} \text{ locally free sheaf on } X \times \operatorname{Spec} A, f : \mathcal{F}|_X \xrightarrow{\cong} \mathcal{E}\}/\{\text{iso}\}.$$

Then, one can show that

(1) The tangent space of $Def_{\mathcal{E}}$ is $T_{\mathcal{E}} = H^1(X, End(\mathcal{E}))$.

(2) The obstruction space of $Def_{\mathcal{E}}$ is $O_{\mathcal{E}} = H^2(X, End(\mathcal{E}))$.

More generally, one could deform a coherent sheaf $\mathcal{E}$ on X. In that case, the tangent space is $Ext^1(\mathcal{E}, \mathcal{E})$ and the obstruction space is $Ext^2(\mathcal{E}, \mathcal{E})$.

Example 5.2. (Deformation of subschemes) Consider the deformation problem of deforming a proper closed subscheme Y of a scheme X over $\mathbb{K}$,

$$Def_{Y\subset X}(A) = \{Z_A \ A\text{-flat closed subscheme of } X \times \operatorname{Spec} A \text{ such that } Z_A \times \mathbb{K} = Z\}.$$

Then, one can show that

(1) The tangent space of $Def_{Y\subset X}$ is $T_{Y\subset X} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{O}_Y)$.

(2) The obstruction space of $Def_{Y\subset X}$ is $O_{Y\subset X} = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{O}_Y)$,

where $\mathcal{I}_Y$ is the ideal sheaf of Y in X.

References

[1] Jacob Lurie. DAG X: Formal moduli problems. available at `www.math.harvard.edu/~lurie/papers/DAG-X.pdf`.

[2] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

[3] Barbara Fantechi and Marco Manetti. Obstruction calculus for functors of Artin rings, I. *Journal of Algebra*, 202(2):541 – 576, 1998.

[4] Brian Osserman. A glimpse of deformation theory. available at `https://www.math.ucdavis.edu/~osserman/classes/256A/notes/deform.pdf`.

Further expository references:

[5] Marco Manetti. Functors of Artin rings, obstructions, and factorization. available at `http://www1.mat.uniroma1.it/people/manetti/DT2011/FuntoriArtin.pdf`.

[6] Brian Osserman. Representability and Schlessinger's criterion. available at `https://www.math.ucdavis.edu/~osserman/math/workshop.pdf`.